Why Frames Tilt Forward: Solution by Craig H Collins

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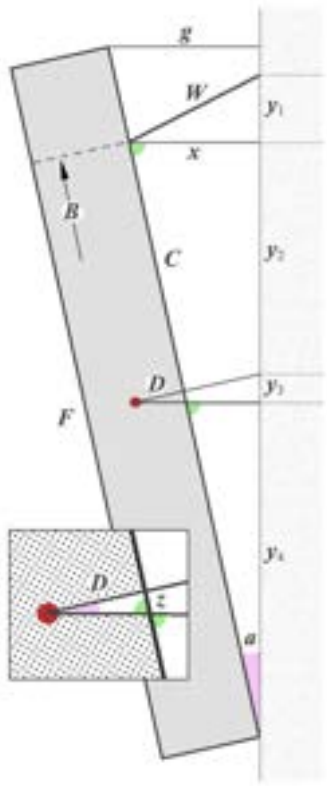

The figure at left shows a frame hanging on a wire attached to a wall. The frame is at equilibrium when its center-of-mass is at the lowest possible position relative to the wall.

In the discussion that follows, known quantities are denoted by upper-case letters and unknowns by lower-case letters. In the figure, angles that are congruent to each other are shown in the same color.

Assume the center-of-mass (red circle) is located $\boldsymbol{D}$ inches from the back of the frame and halfway down the length $\boldsymbol{F}$ of the frame. Also assume that the wire is attached to the frame $\boldsymbol{C}$ inches up from the center of the frame (or $\boldsymbol{B}$ inches up from the bottom) and that the slack in the wire, as viewed from the side, is $\boldsymbol{W}$.

We ultimately want to find $\boldsymbol{g}$, the gap between the top of the frame and the wall, when the center-of-mass is at its lowest (equilibrium) position.

Our first step is to find the equation for the vertical distance $\boldsymbol{y}$ $(= \boldsymbol{y_1} + \boldsymbol{y_2} + \boldsymbol{y_3})$ from the wall attachment point to the center-of-mass, in terms of known quantities $\boldsymbol{F, B, C, D, W}$ and the unknown $\boldsymbol{g}$. Next, we find the derivative of this expression with respect to $\boldsymbol{g}$ and set the derivative to zero to obtain the equilibrium equation. (The derivative is zero when $\boldsymbol{y}$ is at its maximum, i.e., when the center-of-mass is at its lowest point.) Finally, one solves the equilibrium equation for the gap $\boldsymbol{g}$.

(1) We will first find $\boldsymbol{x}$. Note the angle $\boldsymbol{a}$ where the bottom of the frame touches the wall. We do not know $\boldsymbol{a}$, but from trigonometry we know $\sin(\boldsymbol{a}) = \boldsymbol{g/F}$ and $\sin(\boldsymbol{a}) = \boldsymbol{x/B}$. Therefore, $\boldsymbol{x} = \boldsymbol{B}(\boldsymbol{g/F})$.

(2) From Pythagoras' Theorem, $\boldsymbol{W^2} = \boldsymbol{x^2} + \boldsymbol{y_1^2}$. Therefore, $\boldsymbol{y_1} = \sqrt{\boldsymbol{W^2} - \boldsymbol{x^2}} = \sqrt{\boldsymbol{W^2} - \boldsymbol{B^2}(\boldsymbol{g/F})^2}$.

(3) From trigonometry, we know $\cos(\boldsymbol{a}) = (\boldsymbol{y_2} + \boldsymbol{y_3} + \boldsymbol{y_4})/\boldsymbol{B}$. Based on similarity, $\cos(\boldsymbol{a}) = (\boldsymbol{y_2} + \boldsymbol{y_3})/(\boldsymbol{C} + \boldsymbol{z})$. Rearranging gives $\boldsymbol{y_2} + \boldsymbol{y_3} = (\boldsymbol{C} + \boldsymbol{z})\cos(\boldsymbol{a})$.

(4) Since $\cos(\boldsymbol{a}) = \sqrt{1 - \sin^2(\boldsymbol{a})}$ and $\sin(\boldsymbol{a}) = \boldsymbol{g/F}$, then $\boldsymbol{y_2} + \boldsymbol{y_3} = (\boldsymbol{C} + \boldsymbol{z})\sqrt{1 - (\boldsymbol{g/F})^2}$.

(5) That leaves $\boldsymbol{z}$. The angle opposite segment $\boldsymbol{z}$ is congruent to $\boldsymbol{a}$, so $\boldsymbol{z} = \boldsymbol{D}\tan(\boldsymbol{a}) = \boldsymbol{D}\sin(\boldsymbol{a})/\sqrt{1 - \sin^2(\boldsymbol{a})}$ or $\boldsymbol{z} = \boldsymbol{D}(\boldsymbol{g/F})/\sqrt{1 - (\boldsymbol{g/F})^2}$.

(6) Combining (2), (4) and (5) gives $\boldsymbol{y} = \boldsymbol{y_1} + \boldsymbol{y_2} + \boldsymbol{y_3} = \sqrt{\boldsymbol{W^2} - \boldsymbol{B^2}(\boldsymbol{g/F})^2} + \boldsymbol{C}\sqrt{1 - (\boldsymbol{g/F})^2} + \boldsymbol{D}(\boldsymbol{g/F})$.

(7) The derivative of this function with respect to $\boldsymbol{g}$ is

$$\frac{d\boldsymbol{y}}{d\boldsymbol{g}} = -\frac{\boldsymbol{B^2 g}}{\boldsymbol{F^2}\sqrt{\boldsymbol{W^2} - \boldsymbol{B^2}(\boldsymbol{g/F})^2}} - \frac{\boldsymbol{Cg}}{\boldsymbol{F^2}\sqrt{1 - (\boldsymbol{g/F})^2}} + \frac{\boldsymbol{D}}{\boldsymbol{F}}$$

(8) Setting the derivative to 0 and simplifying…

$$\frac{\boldsymbol{B^2 g}}{\sqrt{\boldsymbol{F^2 W^2} - \boldsymbol{B^2 g^2}}} + \frac{\boldsymbol{Cg}}{\sqrt{\boldsymbol{F^2} - \boldsymbol{g^2}}} - \boldsymbol{D} = 0$$

This equation must be solved for $\boldsymbol{g}$ by iteration.